

Equivalence of model space techniques and the renormalization group for a toy 2-body problem

S.K. Bogner and T.T.S. Kuo

Department of Physics, SUNY, Stony Brook, New York 11794, USA

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Abstract

Lee-Suzuki similarity transformations and Krencigowa-Kuo folded diagrams are two common methods used to derive energy independent model space effective interactions for nuclear many-body systems. We demonstrate that these two methods are equivalent to a Renormalization Group (RG) analysis of a toy 2-body problem. This is accomplished by considering a schematic model that can be solved in closed form. Analytic expressions for the effective low-momentum potentials V_{eff} are obtained and shown to give the same scaling equation for V_{eff} that RG arguments predict. This indicates that model space methods should be of interest to those studying low-energy nuclear physics using Effective Field Theories (EFT). We find the new result that all of the different energy independent model space techniques yield a unique low-momentum V_{eff} when applied to the toy 2-body problem.

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I. INTRODUCTION

There has been much effort over the past decade to describe low-energy nuclear phenomena using the techniques of Effective Field Theory (EFT) [1–3,8,5]. The main goal of an EFT description of these phenomena is not to out-perform the traditional methods at fitting the experimental data. Rather it is to provide a *model independent* effective low energy theory whose form is constrained by the symmetries of QCD, as well as to eliminate the uncontrolled and unjustified approximations that are made in the traditional approaches. One of the more appealing features of an EFT treatment is the ability to reliably estimate the errors in calculations. This allows one to consistently calculate observables to any chosen level of accuracy in principle. The success of various low energy effective theories (Landau Fermi Liquid theory, the Fermi theory of Beta decay, the Standard Model, etc) results from the fundamental tenet of EFT: *low energy observables are insensitive to the details of the high energy dynamics*. Even if the high energy dynamics are unknown, one can still mimic their effects on the low energy physics with local, model-independent operators that are consistent with the low-energy symmetries. The corresponding coupling constants then implicitly contain the effects of the integrated out high energy degrees so that the low-energy physics of the full theory is preserved. EFT's are clearly founded upon the ideas first put forward by Wilson in his formulation of the "modern" renormalization group in the study of phase transitions. The early review article of Wilson and Kogut [9] and the more recent papers of Lepage and Polchinski [7,6] illustrate these ideas with pedagogical examples and a minimum of formalism.

Shell-model theorists have long employed the notion of effective interactions that are defined only within a truncated model space, but that implicitly contain the effects of the states that are being thrown away (i.e.- integrated out) so that certain low energy observables are preserved from the full-theory [12–17]. The qualitative similarities between the modern RG techniques and model space methods for deriving effective interactions within a truncated model space are obvious. It would be interesting to study the *quantitative* similarities between the two approaches, as little work seems to have been done in this area. The main purpose of this paper is to study a toy model of the 2-body problem that allows closed form solutions for the low-momentum V_{eff} using both the Lee-Suzuki (LS) and the Krencigowa-Kuo (KK) model space methods. We find that V_{eff} derived from both schemes obeys the same scaling equation that one would obtain from a RG analysis of the 2-body problem [10]. Moreover, we find the new and interesting result that *all* of the various model space techniques corresponding to different methods of solving the decoupling equation (LS and KK are two specific schemes) give the same low-momentum V_{eff} . In other words, the low-momentum V_{eff} is unique and independent of the particular model space scheme one employs for the toy model we consider. This is a surprising result as it is known that when one considers bound state (e.g- shell model) problems the different model space techniques often converge to different (i.e.- non unique) V_{eff} 's [12–17]. It is worth repeating that our V_{eff} is always independent of energy, as this greatly simplifies its use in many-body calculations.

II. MODEL SPACE TECHNIQUES

We turn now to a quick review of the methods one uses to derive energy independent model space effective interactions. Andreozzi has shown that both the Lee-Suzuki (LS) and Kuo-Krencigowa (KK) methods can be recast in the form of a general similarity transformation [16]. In this formalism we perform a similarity transformation to obtain an effective Hamiltonian that acts only within the model space, but that preserves low-energy physics (spectra, scattering amplitudes, etc) from the full problem. If we denote the projection operator onto the model space as P and its complement as Q we have

$$S^{-1}HSS^{-1}|\Psi\rangle = ES^{-1}|\Psi\rangle \quad (1)$$

We can write the above equation in an obvious way by defining $H_{eff} \equiv S^{-1}HS$ and $|\chi\rangle = S^{-1}|\Psi\rangle$. A convenient form of the similarity transformation is $S = 1 + \omega$, where ω is called the wave operator and is defined to satisfy $\omega = Q\omega P$. It is easy to see that $S^{-1} = 1 - \omega$ and that the P-space projections of the transformed and original states are the same, $P|\chi\rangle = P|\Psi\rangle$. We want to exploit the invariance of eigenvalues under similarity transformations to preserve a subset of the exact eigenvalues, but with H_{eff} acting only within the model space. In other words we want $H_{eff} = PH_{eff}P$ and $|\chi\rangle = P|\Psi\rangle$ so that the excluded Q-space completely decouples from the problem. A necessary and sufficient condition for this is $QH_{eff}P = 0$. Explicitly writing this out gives the so-called decoupling equation for ω

$$\omega PHQ\omega + \omega PHP - QHQ\omega - QHP = 0 \quad (2)$$

The decoupling equation is a non-linear operator equation and different methods of solution can give different answers. Andreozzi has recently shown that two iteration methods of solving it are equivalent to the Lee-Suzuki method and the Krencigowla-Kuo folded diagram theory [16]. The differences between the LS and the KK methods are most apparent when one is dealing with bound state problems and the basis states are discrete (e.g.- harmonic oscillator orbits). If the model space is d- dimensional, then the LS H_{eff} reproduces the lowest d eigenvalues of the full problem and the corresponding P-space projections of the exact eigenstates. Conversely, the KK H_{eff} reproduces the d eigenvalues of the full problem whose eigenstates have the largest P-space component (i.e.- maximum overlap). The great benefit of Andreozzi's work is that it contains solutions of the decoupling equation that are formally equivalent to the LS and KK schemes, but his methods are non-perturbative in nature and hence eliminate the need to calculate irreducible vertex functions (i.e.- \hat{Q} -boxes) and perform G-matrix resummations. Andreozzi's methods can furnish analytic results for simple toy models, and yet they are extremely robust and easy to implement numerically for realistic bare potentials (Paris, Bonn-A, V-18, etc...) [11]. In deference to Andreozzi's simplification of KK and LS calculations, we hereafter refer to them as the Andreozzi-Krencigowa-Kuo (AKK) and the Andreozzi-Lee-Suzuki (ALS) schemes. We simply quote the relevant equations and refer the interested reader to Andreozzi's paper for details.

THE ALS METHOD

Writing $\omega = \sum_{n=0}^{\infty} X_n$ and $\sigma_n = \sum_{m=0}^n X_m$, the ALS equations are

$$X_0 = \frac{-1}{QH Q} QHP \quad (3)$$

$$X_n = \frac{1}{q(\sigma_{n-1})} X_{n-1} p(\sigma_{n-1}) \quad n = 1, 2, \dots \quad (4)$$

$$p(\sigma_{n-1}) = PHP + PHQ\sigma_{n-1} \quad (5)$$

$$q(\sigma_{n-1}) = QHQ - \sigma_{n-1}PHQ \quad (6)$$

Note that $p(\sigma_{n-1})$ and $q(\sigma_{n-1})$ are the P-space and Q-space effective Hamiltonians at each step in the iteration, and the iteration converges when $\sigma_n \approx \sigma_{n-1}$.

THE AKK METHOD

Using the same notation as the ALS scheme, the X_n in the AKK scheme are given by the following equations,

$$X_0 PHP - QHQ X_0 - QHP = 0 \quad (7)$$

$$X_n p(\sigma_{n-1}) - QHQ X_n + \sigma_{n-1} PHQ X_{n-1} = 0 \quad n = 1, 2, \dots \quad (8)$$

III. SCHEMATIC MODEL WITH SEPARABLE POTENTIAL

Both the ALS and AKK methods are easy to implement numerically for realistic models of the bare V_{NN} such as the Paris and Bonn-A potentials. However, it would be nice to construct a toy model for which the ALS and AKK equations yield analytic solutions. For this purpose, we consider the following schematic underlying (or full space) Hamiltonian

$$H = H_0 + g \mid \eta \rangle \langle \eta \mid \quad (9)$$

where

$$\langle i \mid H \mid j \rangle = \epsilon_i \delta_{ij} + g \eta_i \eta_j \quad (10)$$

We will use the following formula many times in the analysis that follows.

$$\langle i \mid H^{-1} \mid j \rangle = \frac{\delta_{ij}}{\epsilon_i} - \frac{g \eta_i \eta_j}{\epsilon_i \epsilon_j} \frac{1}{1 + gF} \quad (11)$$

where

$$F = \sum_l \frac{\eta_l^2}{\epsilon_l} \quad (12)$$

Note that we don't have to worry about the above expressions becoming singular ($\epsilon_i = 0$), as we are always inverting in Q-space (where the $\epsilon_i \neq 0$).

IV. ALS AND AKK ANALYSIS OF THE SCHEMATIC MODEL

Since we are analysing the 2-body problem, all state labels refer to relative momenta, and it is assumed we are working with a particular partial wave (labels are suppressed). If we assume that the underlying potential $V(k', k) \approx 0$ for $k, k' > \Lambda$ (either by a form factor or a sharp cut-off), then the truncated model space will consist of all states with $k \leq \frac{\Lambda}{s}$, where $s > 1$. The excluded Q-space will then consist of all states with momenta lying in the shell $\frac{\Lambda}{s} < k \leq \Lambda$. For a finite shrinking or decimation (e.g. $s = 2$), we can use the formulas of the previous two sections to calculate the X_n of both the ALS and AKK schemes analytically. Unfortunately the expressions quickly become unwieldy for $n > 1$, and it is not at all obvious how one can sum all of the X_n to obtain an analytic expression for the wave operator (and hence for V_{eff}). The trick that gets us off the hook is to consider infinitesimal decimations where $\frac{\Lambda}{s} = \Lambda - \delta\Lambda$. In this way we are able to calculate the wave operator analytically and obtain a scaling equation for V_{eff} . Using the definition $V_{eff} = H_{eff} - H_0$, cavalierly ignoring factors of 2π , and working in units where $\epsilon_p = p^2$, we have

$$V_{eff}(k', k) = V(k', k) + \int_{\Lambda - \delta\Lambda}^{\Lambda} V(k', q) \omega(q, k) q^2 dq \quad k, k' \leq \Lambda - \delta\Lambda \quad (13)$$

For $\delta\Lambda \rightarrow 0$, this becomes a flow equation for V_{eff}

$$\frac{\partial}{\partial \Lambda} V_{eff}(k', k) = -\Lambda^2 V_{eff}(k', \Lambda) \omega(\Lambda, k) \quad (14)$$

Since we take the limit of $\delta\Lambda \rightarrow 0$, we can safely ignore contributions to ω that are of $O(\delta\Lambda)$ and higher. With this in mind, both the ALS and the AKK equations simplify considerably.

ALS FOR INFINITESIMAL DECIMATIONS

Recalling that each intermediate projection operator $Q = \int_{\Lambda - \delta\Lambda}^{\Lambda} q^2 |q\rangle \langle q| dq \sim O(\delta\Lambda)$, we can replace all of the $p(\sigma_{n-1}) = PHP + PHQ\sigma_{n-1}$ by PHP in the ALS equations. It can be shown that at each step in the ALS iteration scheme we can write

$$\langle q' | q(\sigma_{n-1}) | q \rangle = \epsilon_q \delta_{q'q} + g \bar{\eta}_{q'} \eta_q \quad (15)$$

Where $\bar{\eta}_{q'} = \eta_{q'}(1 + \text{constants})$. Consequently, we can use equations 11 and 12 to write

$$\langle q' | q^{-1}(\sigma_{n-1}) | q \rangle = \frac{\delta_{q'q}}{\epsilon_q} - \frac{g \bar{\eta}_{q'} \eta_q}{\epsilon'_q \epsilon_q} + O(\delta\Lambda) \quad (16)$$

Naively, one might think that the X_n are all infinitesimals of $O(\delta\Lambda)$ since the ALS equations all have an intermediate Q projection operator. *This is wrong.* One must remember that by equation 16, the operators $(QHQ)^{-1}$ and $q^{-1}(\sigma_{n-1})$ consist of a diagonal part proportional to a δ -function plus an off-diagonal part. It is clear that when an intermediate infinitesimal projection operator Q acts on the off-diagonal part it results in a term of $O(\delta\Lambda)$ that can be dropped. However, the δ -function portion survives the infinitesimal Q projection and

gives a finite contribution. Therefore, we can make a second major simplification to the ALS equations by replacing $(QHQ)^{-1}$ and $q^{-1}(\sigma_{n-1})$ by $(QH_0Q)^{-1}$. The simplified ALS equations can be solved by addition, resulting in the following linear integral equation for ω

$$\omega = X_0 + \frac{1}{QH_0Q} \omega PHP \quad (17)$$

Rearranging terms, explicitly displaying indices, and abbreviating integrals as sums, we obtain

$$\sum_{p'}^{\Lambda-\delta\Lambda} \omega(\Lambda, p') A(p', p) = -\epsilon_\Lambda X_0(\Lambda, p) \quad (18)$$

Where we have defined the matrix $A(p', p) = (\epsilon_{p'} - \epsilon_\Lambda) \delta_{p'p} + g\eta_{p'}\eta_p$. Multiplying from the right by A^{-1} we obtain

$$\omega(\Lambda, p) = -\epsilon_\Lambda \sum_{p'} X_0(\Lambda, p') A^{-1}(p', p) \quad (19)$$

Once again, we make use of equation 11 to obtain an explicit expression for A^{-1} , viz,

$$A^{-1}(p', p) = \frac{\delta_{p'p}}{\epsilon_p - \epsilon_\Lambda} - \frac{g\eta_{p'}\eta_p}{(\epsilon_p - \epsilon_\Lambda)(\epsilon_{p'} - \epsilon_\Lambda)} \frac{1}{1 + g\bar{F}} \quad (20)$$

where we have defined \bar{F} by the following equation

$$\bar{F} = \int_0^{\Lambda-\delta\Lambda} \frac{p^2 \eta_p^2}{p^2 - \Lambda^2} dp \quad (21)$$

Since we eventually take $\delta\Lambda \rightarrow 0$, it is clear that $\bar{F} \rightarrow \infty$ allowing us to ignore the second term in the above expression for A^{-1} . Noting that $X_0(\Lambda, p) = \frac{-g\eta_\Lambda\eta_p}{\epsilon_\Lambda}$ and utilizing the simplified expression for A^{-1} , we obtain our final expressions for the ALS wave operator $\omega(\Lambda, p)$ and the ALS flow equation

$$\omega(\Lambda, p) = \frac{-g\eta_\Lambda\eta_p}{\Lambda^2 - p^2} = \frac{-V_{eff}(\Lambda, p)}{\Lambda^2 - p^2} \quad (22)$$

and

$$\frac{\partial}{\partial\Lambda} V_{eff}(p', p) = \frac{\Lambda^2 V_{eff}(p', \Lambda) V_{eff}(\Lambda, p)}{\Lambda^2 - p^2} \quad (23)$$

AKK FOR INFINITESIMAL DECIMATIONS

For infinitesimal model space reductions $\Lambda \rightarrow \Lambda - \delta\Lambda$, we simply state the simplified AKK equations as we use the same arguments we used to simplify the ALS equations in the previous section. Equation 7 simplifies to

$$X_0 PHP - QH_0 QX_0 = QHP \quad (24)$$

while equation 8 simplifies to

$$X_n PHP - QH_0 QX_n = 0 \quad n = 1, 2, \dots \quad (25)$$

It is easily shown that all of the X_n are zero except for X_0 . Hence, the AKK wave operator is given by X_0 and it is found to be the same as the ALS wave operator. We can generalize the above results to cover *any* method of solving the decoupling equation, by once again considering an infinitesimal $\Lambda \rightarrow \Lambda - \delta\Lambda$. In this case the non-linear decoupling equation simplifies to

$$\omega PHP - QH_0 Q\omega - QHP = 0 \quad (26)$$

It is easily shown that the solution to this equation is the same as the ALS and AKK solutions. Hence, when we shrink the boundary of P-space by an infinitesimal amount, all of the various methods of solving the decoupling equation give the same equation for $\partial_\Lambda V_{eff}$. We shall find that we obtain the same scaling equation if we cut off all loop integrals in the Lippman-Schwinger equation and demand the observable scattering amplitude be independent of Λ ; this is the RG method described in the next section. We can now invoke the semi-group property of the RG and make a much stronger statement about our model space techniques. Even for *finite* $\Lambda \rightarrow \frac{\Lambda}{s}$, the different methods of solving the decoupling equation give a unique low-momentum V_{eff} for the toy 2-body problem under consideration. To the best of our knowledge, this is a new and interesting result. This result brings out yet another similarity between model space techniques and the Wilsonian RG. In the application of the Wilsonian RG to critical phenomena, there are many different methods of integrating out the short distance physics to obtain an effective long-wavelength theory. For example, there are infinitely many prescriptions for performing block spin transformations and assigning values to the block variables for the Ising model. However, the differences between the various coarse graining procedures are eventually washed out and one always converges to the same long-wavelength effective Hamiltonian (i.e.-fixed point).

V. RENORMALIZATION GROUP ANALYSIS

Since we claim that the model space techniques of nuclear many-body physics give the same scaling equation for V_{eff} that a renormalization group analysis predicts, it is convenient to quickly review exactly what we mean by a "renormalization group analysis". In relativistic quantum field theory when one calculates physical observables perturbatively as a power series in the coupling g , one finds that the coefficients of the higher order powers of g are divergent. These infinities are the result of loop integrals that diverge at high momentum, or equivalently at short distances. One can isolate these infinities by regulating the divergent integrals with a UV cut-off or any other convenient method (dimensional regularization, Pauli-Villars, etc...), but then one is faced with the unpleasant result that calculated observables depend on the arbitrary regulator mass Λ that has been introduced into the problem, as well as the particular method of regularization. As is well known, the way out of this dilemma is to notice that for certain field theories (the so-called "renormalizable" theories) we can add counterterms to the theory that have the same operator

structure as the terms present in the initial Lagrangian. If we are sufficiently clever, we can hide the regulated infinities in the coefficients of the counterterms so that when we calculate graphs using the new Feynman rules (that are induced by the counterterms), everything is finite. Hence, one can absorb the divergences by redefining the couplings and making them Λ -dependent in such a way that the *physical observables* are Λ -independent and finite as $\Lambda \rightarrow \infty$. Thus, a change in the UV cut-off scale is compensated by a simultaneous change in the couplings and masses leaving the physics invariant. This is the essence, albeit a grotesquely simplified description, of the renormalization group in QFT designed to hide the infinities at short distances. But by allowing $\Lambda \rightarrow \infty$ we are assuming that the dynamics encoded in the original Lagrangian holds true at arbitrarily large energies. In the language of old fashioned perturbation theory, allowing loop momenta to go up to infinity corresponds to summing over intermediate states of arbitrarily large energy and momentum. But this presupposes that the intermediate states we are summing are devoid of any "new" physics at arbitrarily small distances, which is clearly a bold assumption (supersymmetric partners? quark and lepton substructure? strings?) since $1 \text{ TeV} \neq \infty$! About the only thing we can say for certain about the high energy intermediate states is that they are highly virtual and thus localized in space-time.

The modern approach to renormalization views Λ as a physical scale at which unknown physics comes into play, and not as an artifact that is to be taken to infinity at the end of the calculation. In this way we don't sum over intermediate states we don't fully understand. Yet we can model the effects of these excluded states on the low energy physics by recalling that their virtual nature means that they are propagated over very small distances and times (i.e.- they are localized). We can include their effects on the low energy physics by writing down every possible local interaction that is consistent with low energy symmetries (model independence!), and we can parameterize our ignorance of the the excluded high energy states inside of the tunable couplings that multiply each local operator. Although there are an infinite number of local operators consistent with the symmetry requirements, they all scale as inverse powers of Λ allowing one to truncate the effective Lagrangian to any chosen level of accuracy. The corresponding couplings can then be obtained from a finite number of independent experimental measurements, giving the theory predictive power to the chosen level of accuracy. In this way the effective theory is consistent with the underlying symmetries, the errors are reliably estimated, different quantities can be calculated consistently to the same level of accuracy, and one can infer the presence of "new physics" when the effective theory fails!

We can employ similar ideas in analysing our toy 2-body potential model. Here, we can impose a UV cut-off Λ in the loop integrals of the Lipmann-Schwinger equation to reflect our ignorance of what is really going on at short distances. We then demand that V_{eff} depend on Λ in such a way that the low energy observables (i.e.- the scattering amplitudes) are independent of Λ . For our separable toy model we can write the half-onshell (HOS) T-matrix as

$$\frac{1}{\langle k' | T(\omega = k^2) | k \rangle} = \frac{1}{g\eta_{k'}\eta_k} - \frac{1}{\eta_{k'}\eta_k} \int_0^\Lambda \frac{q^2 \eta_q^2}{k^2 - q^2} dq \quad (27)$$

Since we want the physical quantities to be independent of this UV cut-off Λ , we allow the coupling to "run" with Λ . By setting $\partial_\Lambda T^{-1} = 0$ (and noting that all the Λ -dependence is

inside g), we obtain a scaling equation for V_{eff}

$$\frac{\partial}{\partial \Lambda} V_{eff}(k', k) = \frac{\Lambda^2 V_{eff}(k', \Lambda) V_{eff}(\Lambda, k)}{\Lambda^2 - k^2} \quad (28)$$

This is same scaling equation for V_{eff} that we derived using model space methods! We also note that this is the same equation obtained by Birse et. al. [10], except that their V_{eff} depends on the energy as well (i.e. $V_{eff}(k', k) \rightarrow V_{eff}(k', k; E)$). We therefore conclude that model space techniques preserve the low energy HOS T-matrix and are equivalent to renormalization group methods when applied to the 2-body problem.

VI. CONCLUSIONS

We have demonstrated that the various model space methods used to construct energy independent effective interactions are equivalent to renormalization group methods. This was accomplished by considering a separable 2-body potential model in which the model space techniques yield analytic expressions for V_{eff} . We have found that model space methods and renormalization group techniques give the same scaling equation for V_{eff} . Moreover, we have found that all methods of solving the decoupling equation give the same V_{eff} for the toy model under consideration. To the best of our knowledge, these are new and interesting results. The generalization of these results to realistic V_{NN} 's will appear in a forthcoming paper.

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